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Multivariate time changes for Lévy asset models: Characterization and calibration[☆]

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ABSTRACT

We build a theoretical framework for multivariate subordination of Brownian motions, with a common and an idiosyncratic component. This follows economic intuition and introduces generalizations of some well known multivariate Lévy processes for financial applications: the compound Poisson, NIG, Variance Gamma and CGMY. In most cases we obtain the characteristic function in closed form. The extension is first kept parsimonious, by adding one parameter only. The empirical fit of (linear) dependence is then increased, by allowing for dependent Brownian motions.

0. Introduction

There are theoretical as well as empirical reasons for being interested in time changes.

On the theoretical side, price processes under no arbitrage are semimartingales. The latter can be represented as time changed Brownian motions, using as time change either a subordinator or a more general process. When the change of time is a subordinator the resulting process belongs to the (pure jump) Lévy class. This explains the success of subordination in mathematical Finance, in order to represent asset prices.

On the empirical side time changes model the flow of information, as measured by trade: one can think that time runs fast when there are a lot of orders, while it slows down when trade is stale. Economic time then does not coincide with calendar time. This relationship between price changes and trade has been extensively studied (see for instance [1,2]). It has been tested in Geman and Ané [3], which cannot reject normality of re-scaled returns, i.e. returns per unit of trade.

Most of the time change literature considers one asset at a time.

At the multivariate level, time changing has been studied much less. Multivariate Lévy processes have been generally constructed subordinating a multivariate Brownian motion by means of a univariate subordinator. Such processes present a number of drawbacks, including restrictions on the marginal parameters, lack of independence and possibility of spanning a limited range of dependence (see for instance [4]). On top of that, a unique time change for all assets does not seem to be economically sound. Harris [5], while discussing the cross sectional properties of trades, rejects equality across different

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assets. Using an extensive data set from the US stock market, Lo and Wang [6] show that trades present a significant common component.

Multivariate subordinators, i.e. different time changes for different assets, appeared only very recently: the theoretical set up is in [7]. Eberlein and Madan [8] introduce a multivariate subordinator with independent components to time change dependent Brownian motions and fit financial returns.

The multivariate subordinator studied here incorporates both a time transform common to all assets and an idiosyncratic one. The former can be interpreted in financial applications as a measure of the overall trade or market activity, while the latter represents asset-specific trade. The empirical analysis performed in [6] justifies our choice.

We build on the idea of splitting the multivariate subordinator into a common and an idiosyncratic component, which we introduced for the Variance Gamma (VG) case in [9] and for the generalized hyperbolic (GH) one in [10]. We extend the previous papers as follows. We first characterize the change of time, then the corresponding time changed Brownian motions. We introduce generalizations of the multivariate Compound Poisson (CP), Normal Inverse Gaussian (NIG), Variance Gamma and Carr Geman Madan Yor (CGMY) processes. In order to justify our modelling choices and their usefulness for financial applications, we have three desired features in mind: the existence of characteristic functions in closed form, the ability to capture a wide range of (linear dependence) and the possibility of calibrating marginal and joint parameters separately.

We first provide the characteristic functions, for all cases except CGMY.

We then study the nonlinear and linear dependence of the processes so obtained. We show that, as long as our multivariate subordinator is applied to independent Brownian motions, the model is extremely parsimonious in terms of parameters: there is only one additional parameter on top of the marginal ones. The range of linear correlation which can be captured is bounded above by the change of time (trade) correlation. Therefore, we extend the model so as to incorporate correlated Brownian motions. The number of parameters needed for calibration increases, but the upper bound on correlation disappears.

Last but not least, we provide a calibration technique, which shows that one can fit separately the marginal distributions and the correlation parameters. Consequently, one can shift from the more to the less parsimonious model, in order to improve the correlation fit, without re-estimating the marginal distributions.

The paper proceeds as follows: Section 1 contains some basic terminology. Section 2 presents our class of multivariate subordinators. Section 3 applies them to independent Brownian motions and obtains the general properties (Lévy nature, characteristic function, Lévy triplet and measure) of the corresponding subordinated processes. The results are then specified in relation to the CP, NIG, VG and CGMY cases. Section 4 studies dependence. Section 5 extends the model to correlated Brownian motions. In view of the empirical applications, Section 6 discusses linear correlation. Section 7 contains a calibration for seven major stock indices. Section 8 concludes.

1. Preliminaries

Let $\boldsymbol{X}(t)$ be a $\mathbb{R}^n$-valued Lévy process. The characteristic function is fundamental in its construction. It admits the following *Lévy–Khinchin representation*:

$$\psi_{\boldsymbol{X}(t)}(\boldsymbol{z}) = E[\mathrm{e}^{\mathrm{i}\langle \boldsymbol{z}, \boldsymbol{X}(t)\rangle}] = \mathrm{e}^{t\Psi_{\boldsymbol{X}}(\boldsymbol{z})}, \quad \boldsymbol{z} \in \mathbb{R}^n,$$

with

$$\Psi_{\boldsymbol{X}}(\boldsymbol{z}) = -\frac{1}{2}\langle \boldsymbol{z}, A\boldsymbol{z}\rangle + \mathrm{i}\langle \boldsymbol{\gamma}, \boldsymbol{z}\rangle + \int_{\mathbb{R}^n} (\mathrm{e}^{\mathrm{i}\langle \boldsymbol{z}, \boldsymbol{x}\rangle} - 1 - \mathrm{i}\langle \boldsymbol{z}, \boldsymbol{x}\rangle 1_{|\boldsymbol{x}|\le 1})\nu(\mathrm{d}\boldsymbol{x}),$$

where A is a symmetric $n \times n$ matrix, $\boldsymbol{\gamma} \in \mathbb{R}^n$ and ν is a positive random measure on $\mathbb{R}^n$. $(\boldsymbol{\gamma}, A, \nu)$ is called the Lévy triplet of the process. $\Psi_{\boldsymbol{X}}(z)$ is named characteristic exponent of $\boldsymbol{X}$. In the next section we focus our attention on a particular class of Lévy processes, the subordinators, that are increasing Lévy processes. They have no diffusion component and are of finite variation. More precisely, we are interested in the multivariate generalization of subordinators. A *multivariate subordinator* is a Lévy process on $\mathbb{R}^n_+ = [0, \infty)^n$, whose trajectories are increasing in each coordinate. See [7] for the main properties of such processes. The characteristic exponent of a multivariate subordinator has the following expression:

$$\Psi_{\boldsymbol{X}}(\boldsymbol{z}) = \mathrm{i}\langle \boldsymbol{m}, \boldsymbol{z}\rangle + \int_{\mathbb{R}^n} (\mathrm{e}^{\mathrm{i}\langle \boldsymbol{z}, \boldsymbol{x}\rangle} - 1)\nu(\mathrm{d}\boldsymbol{x}), \tag{1.1}$$

where $\boldsymbol{m} \in \mathbb{R}^n_+$ and $\nu_{\boldsymbol{X}}$ is the Lévy measure of $\boldsymbol{X}$.

A theorem that plays a central role in the characterization in terms of Lévy triplet of the processes we are going to construct is in [7]. The theorem requires the introduction of the multi-parameter process notion. Consider n real independent Lévy processes $X_1(t), \ldots, X_n(t)$. The stacked process $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))^{\mathrm{T}}$, where the superscript T denotes the transpose, is then a Lévy process on $\mathbb{R}^n$. Consider the multi-parameter $\boldsymbol{s} = (s_1, \ldots, s_n)^{\mathrm{T}} \in \mathbb{R}^n_+$ and the partial order on $\mathbb{R}^n_+$:

$$\boldsymbol{s}^1 \preceq \boldsymbol{s}^2 \Leftrightarrow s^1_j \le s^2_j, \quad j = 1, \ldots, n.$$

Define the multi-parameter process $\{\boldsymbol{X}(\boldsymbol{s}), \boldsymbol{s} \in \mathbb{R}^n_+\}$ by

$$\boldsymbol{X}(\boldsymbol{s}) = (X_1(s_1), \ldots, X_n(s_n))^{\mathrm{T}}.$$

Theorem 1.1. *Let $\boldsymbol{G}$ be a multivariate subordinator with triplet $(\boldsymbol{\gamma_G}, 0, \nu_{\boldsymbol{G}})$ and let $\lambda_t = \mathcal{L}(\boldsymbol{G}(t))$. Let $\boldsymbol{X}(t)$ be a Lévy process on $\mathbb{R}^n$, independent from $\boldsymbol{G}$, with independent components and triplet $(\boldsymbol{\gamma_X}, \Sigma_{\boldsymbol{X}}, \nu_{\boldsymbol{X}})$, where $\Sigma_{\boldsymbol{X}} = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$, and let $\rho_{\boldsymbol{s}} = \mathcal{L}(X(\boldsymbol{s}))$. Define the process $\boldsymbol{Y} = \{\boldsymbol{Y}(t), t \geq 0\}$ by the following*

$$\boldsymbol{Y}(t) = (X_1(G_1(t)), \ldots, X_n(G_n(t)))^{\mathrm{T}}, \quad t \geq 0$$

then the process $\boldsymbol{Y}$ is a Lévy process and

$$E[\mathrm{e}^{\mathrm{i}\langle \boldsymbol{z}, \boldsymbol{Y}(t)\rangle}] = \exp(t\Psi_{\boldsymbol{G}}(\log \psi_{\boldsymbol{X}}(\boldsymbol{z}))), \quad \boldsymbol{z} \in \mathbb{R}^n_+,$$

where, for any $\boldsymbol{w} = (w_1, \ldots, w_n)^{\mathrm{T}} \in \mathbb{C}^n$ with $\mathrm{Re}\,(w_j) \leq 0, j = 1, \ldots, n$, we let

$$\Psi_{\boldsymbol{G}}(\boldsymbol{w}) = \langle \boldsymbol{m} \cdot \boldsymbol{w}\rangle + \int_{\mathbb{R}^n} (\mathrm{e}^{\langle \boldsymbol{w}, \boldsymbol{x}\rangle} - 1)\nu(\mathrm{d}\boldsymbol{x}).$$

Moreover the characteristic triplet $(\boldsymbol{\gamma_Y}, \Sigma_{\boldsymbol{Y}}, \nu_{\boldsymbol{Y}})$ of $\boldsymbol{Y}$ is as follows

$$\begin{aligned}
\boldsymbol{\gamma_Y} &= \int_{\mathbb{R}^n_+} \nu_{\boldsymbol{G}}(\mathrm{d}s) \int_{|\boldsymbol{x}|\leq 1} \boldsymbol{x}\rho_{\boldsymbol{s}}(\mathrm{d}\boldsymbol{x}) + \langle \boldsymbol{m}, \boldsymbol{\gamma_X}\rangle,\\
\Sigma_{\boldsymbol{Y}} &= \mathrm{diag}(m_1\sigma_1, \ldots, m_n\sigma_n)\\
\nu_{\boldsymbol{Y}}(\boldsymbol{B}) &= \nu_1(\boldsymbol{B}) + \nu_2(\boldsymbol{B})
\end{aligned}$$

where ν_1 and ν_2 are defined by $\nu_1(\boldsymbol{0}) = 0$, $\nu_2(\boldsymbol{0}) = 0$ and – for $\boldsymbol{B} \in \mathcal{B}(\mathbb{R}^n \setminus \boldsymbol{0})$ – by

$$\begin{aligned}
\nu_1(\boldsymbol{B}) &= \int_{\mathbb{R}^n_+} \rho_{\boldsymbol{s}}(\boldsymbol{B})\nu_{\boldsymbol{G}}(\mathrm{d}\boldsymbol{s}),\\
\nu_2(\boldsymbol{B}) &= \int_{\boldsymbol{B}} m_1 1_{A_1}(x)\nu_{X_1}(\mathrm{d}x) + \cdots + m_n 1_{A_n}(x)\nu_{X_n}(\mathrm{d}x),
\end{aligned}$$

where $x \in \mathbb{R}$, ν_{X_i}, $i = 1, \ldots, n$ are the Lévy measures of the independent marginal processes of $\boldsymbol{X}$ and $A_i = \{\boldsymbol{x} = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n : x_k = 0 \text{ for } k \neq i, k = 1, \ldots, n\}$, $i = 1, \ldots, n$. If $\boldsymbol{m} = \boldsymbol{0}$ and $\int_{|\boldsymbol{s}|\leq 1} |\boldsymbol{s}|^{\frac{1}{2}} \nu_{\boldsymbol{G}}(\mathrm{d}\boldsymbol{s}) < \infty$, then $\Sigma_{\boldsymbol{Y}} = \boldsymbol{0}$, $\int_{|\boldsymbol{x}|\leq 1} |\boldsymbol{x}|\nu(\mathrm{d}\boldsymbol{x}) < \infty$, $\boldsymbol{Y}$ has zero drift and is of bounded variation on any time interval almost surely.

2. A class of multivariate subordinators

In this section we introduce a class of multivariate time changes: each one is a sum of an idiosyncratic and a common component. A similar construction has been proposed in [9] for the VG case. Since through these changes of time we aim at obtaining Lévy processes, we assume that they are subordinators.

We define a multidimensional subordinator as follows: let $X_j = \{X_j(t), t \geq 0\}$, $j = 1, \ldots, n$ and $Z = \{Z(t), t \geq 0\}$ be independent subordinators. The Lévy processes $\boldsymbol{G} = \{\boldsymbol{G}(t), t \geq 0\}$ defined by:

$$\boldsymbol{G}(t) = (X_1(t) + \alpha_1 Z(t), \ldots, X_n(t) + \alpha_n Z(t)), \quad \alpha_i > 0,\ i = 1, \ldots, n \tag{2.1}$$

is a multidimensional subordinator. If we denote by Ψ_j and Ψ_Z respectively the characteristic exponents of the processes X_j, $j = 1, \ldots, n$ and Z, namely

$$\begin{aligned}
\Psi_j(w) &= \int_{\mathbb{R}_+} (\mathrm{e}^{\mathrm{i}wz} - 1)\nu_j(\mathrm{d}z) + \mathrm{i}l_j w, \quad j = 1, \ldots, n\\
\Psi_Z(w) &= \int_{\mathbb{R}_+} (\mathrm{e}^{\mathrm{i}wz} - 1)\nu_Z(\mathrm{d}z) + \mathrm{i}l_z w, \quad w \in \mathbb{C},\ w \geq 0,
\end{aligned} \tag{2.2}$$

then the characteristic exponent $\Psi_{\boldsymbol{G}}$ of $\boldsymbol{G}$ satisfies:

$$\begin{aligned}
\Psi_{\boldsymbol{G}}(\boldsymbol{w}) &= \sum_{j=1}^n \Psi_j(w_j) + \Psi_Z\left(\sum_{j=1}^n \alpha_j w_j\right)\\
&= \sum_{j=1}^n \int_{\mathbb{R}^+} (\mathrm{e}^{\mathrm{i}w_j z_j} - 1)\nu_j(\mathrm{d}z) + \mathrm{i}(l_j w_j) + \int_{\mathbb{R}^+} \left(\mathrm{e}^{\mathrm{i}\left(\sum_{j=1}^n \alpha_j w_j\right)s} - 1\right)\nu_Z(\mathrm{d}s) + \mathrm{i}\left(l_z\left(\sum_{j=1}^n \alpha_j w_j\right)\right),
\end{aligned} \tag{2.3}$$

for any $\boldsymbol{w} = (w_1, \ldots, w_n)^{\mathrm{T}} \in \mathbb{C}^n$ with $Re(w_j) \leq 0, j = 1, \ldots, n$. Observe that if X_j, $j = 1, \ldots, n$ and Z have zero drift, so does $\boldsymbol{G}$. Throughout the paper the subordinators we are going to consider for X_j, $j = 1, \ldots, n$ and Z will have zero drift.

Remark 1. The random variables $X_i(1)$, $i = 1, \dots, n$ and $Z(1)$ are independent and infinitely divisible random variables, with characteristic functions respectively ψ_i, $i = 1, \dots, n$ and ψ_Z. Therefore the random vector $\boldsymbol{W}$:

$$\boldsymbol{W} = (W_1, W_2, \dots, W_n)^{\mathrm{T}} = (X_1(1) + \alpha_1 Z(1), X_2(1) + \alpha_2 Z(1), \dots, X_n(1) + \alpha_n Z(1))^{\mathrm{T}}, \tag{2.4}$$

where $\alpha_j, j = 1, \dots, n$ are positive parameters, is jointly infinitely divisible and its characteristic function, $\psi_{\boldsymbol{W}}$, is:

$$\psi_{\boldsymbol{W}}(u_1, u_2, \dots u_n) = \prod_{j=1}^{n} \psi_j(u_j)\psi_Z\left(\sum_{j=1}^{n} \alpha_j u_j\right). \tag{2.5}$$

Let $\tilde{\boldsymbol{G}} = \{\tilde{\boldsymbol{G}}(t), t \geq 0\}$ be the Lévy process which law at time one is $\mathcal{L}(\boldsymbol{W})$, i.e. $\mathcal{L}(\tilde{\boldsymbol{G}}(1)) = \mathcal{L}(\boldsymbol{W})$. Semeraro [9] proved that $\mathcal{L}(\tilde{\boldsymbol{G}}(1)) = \mathcal{L}(\boldsymbol{G}(1))$. Therefore $\boldsymbol{G}$ and $\tilde{\boldsymbol{G}}$ are the same subordinator in law.

3. Time change for independent Brownian motions

We are ready to use the multivariate subordinators above in order to time change independent Brownian motions.

Let $B_j = \{B_j(t), t \geq 0\}$ $j = 1, \dots, n$ be independent standard Brownian motions. Consider the process $\boldsymbol{B} = \{\boldsymbol{B}(t), t > 0\}$

$$\boldsymbol{B}(t) = (\mu_1 t + \sigma_1 B_1(t), \dots, \mu_n t + \sigma_n B_n(t))^{\mathrm{T}}, \quad \mu_i \in \mathbb{R},\ \sigma_i \in \mathbb{R}_+ \setminus \{0\}. \tag{3.1}$$

The Lévy triplet of $\boldsymbol{B}$ is obviously $(\boldsymbol{\mu}, \Sigma, 0)$, where $\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_n)$ $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$.

The $\mathbb{R}^n$-valued time changed process $\boldsymbol{Y} = \{\boldsymbol{Y}(t), t > 0\}$ is defined as:

$$\boldsymbol{Y}(t) = \begin{pmatrix} \mu_1 G_1(t) + \sigma_1 B_1(G_1(t)) \\ \cdots \\ \mu_n G_n(t) + \sigma_n B_n(G_n(t)) \end{pmatrix}, \tag{3.2}$$

where $\boldsymbol{G}$ is a multivariate subordinator defined by (2.1), independent from $\boldsymbol{B}$. The time changed processes will be interpreted as log returns or log prices: $\boldsymbol{Y}(t) = \log \boldsymbol{S}(t)$ where $\boldsymbol{S}(t)$ collects the time t prices of n assets.

The process $\boldsymbol{Y}$, as given by (3.2), is a Lévy process with characteristic function

$$E[\mathrm{e}^{\mathrm{i}\langle \boldsymbol{z}, \boldsymbol{Y}(t)\rangle}] = \exp(t\Psi_{\boldsymbol{G}}(\log \psi_{\boldsymbol{B}}(\boldsymbol{z}))), \quad \boldsymbol{z} \in \mathbb{R}^n_+,$$

where $\psi_{\boldsymbol{B}}$ is the characteristic function of the Brownian motion $\boldsymbol{B}$ and $\Psi_{\boldsymbol{G}}$ is the characteristic exponent of $\boldsymbol{G}$ (see (2.3)). Observe that the process $\boldsymbol{Y}$ is pure jump.

Using Theorem 1.1 we can state that the characteristic triplet $(\boldsymbol{\gamma}_{\boldsymbol{Y}}, \Sigma_{\boldsymbol{Y}}, \nu_{\hat{\boldsymbol{Y}}})$ of $\boldsymbol{Y}$ is as follows

$$\begin{aligned} \boldsymbol{\gamma}_{\boldsymbol{Y}} &= \int_{\mathbb{R}^n_+} \nu_{\boldsymbol{G}}(\mathrm{d}\boldsymbol{s}) \int_{|\boldsymbol{x}| \leq 1} \boldsymbol{x} \rho_{\boldsymbol{s}}(\mathrm{d}\boldsymbol{x}), \\ \Sigma_{\boldsymbol{Y}} &= \boldsymbol{0}, \\ \nu_{\boldsymbol{Y}}(B) &= \int_{\mathbb{R}^n_+} \rho_{\boldsymbol{s}}(B)\nu_{\boldsymbol{G}}(\mathrm{d}\boldsymbol{s}), \end{aligned} \tag{3.3}$$

where ρ_s is the law of $\boldsymbol{B}(\boldsymbol{s})$ (shortly $\rho_s = \mathcal{L}(\boldsymbol{B}(\boldsymbol{s}))$), $s \in \mathbb{R}^n_+$ and $B \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$.

Starting from the previous theorem we can also discuss the regularity of the trajectories of the process $\boldsymbol{Y}$, namely its finite/infinite activity and its bounded/unbounded variation.

As concerns the activity, an immediate consequence of

$$\nu_{\boldsymbol{Y}}(\mathbb{R}^d) = \int_{\mathbb{R}^n_+} \rho_{\boldsymbol{s}}(\mathbb{R}^d)\nu_{\boldsymbol{G}}(\mathrm{d}\boldsymbol{s}) = \int_{\mathbb{R}^n_+} \nu_{\boldsymbol{G}}(\mathrm{d}\boldsymbol{s}) = \nu_{\boldsymbol{G}}(\mathbb{R}^n), \tag{3.4}$$

is that $\boldsymbol{Y}$ has finite activity ($\nu_{\boldsymbol{Y}}(\mathbb{R}^d) < \infty$) if and only if $\boldsymbol{G}$ does ($\nu_{\boldsymbol{G}}(\mathbb{R}^d) < \infty$). We can also infer the path regularity of the process as a whole from its marginal properties, as follows. The marginal Lévy measures are defined by

$$\nu_j(A) = \nu_{\boldsymbol{Y}}(\mathbb{R} \times A_j \cdots \times \mathbb{R}), A_j \in \mathcal{B}(\mathbb{R}), \quad j = 1, \dots, n. \tag{3.5}$$

It follows that $\nu_j(\mathbb{R}) < \infty$ for all $j = 1, \dots, n$ iff $\nu(\mathbb{R}^n) < \infty$.

As concerns the variations, $\boldsymbol{Y}$ has finite variations if and only if the margins do.[1]

We now discuss different specifications of the $\boldsymbol{Y}$ process. They are multivariate generalizations of log prices models widely studied in Finance. The main properties of the corresponding univariate versions are recalled in Appendix A. Here we simply recall how the univariate versions can be built via a change of time.

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[1] The paths of $\boldsymbol{Y}$ are vectorial functions whose components are the paths of its marginal processes. Therefore the previous statement is a consequence of the fact that a vectorial function has bounded variation (has finite length) if and only if its components have bounded variations.

3.1. Compound Poisson margins

Geman, Madan, Yor [11] proved that the Poisson model with reflected normal jump intensity can be constructed by Poisson time changing a univariate Brownian motion. Consider the univariate compound Poisson process:

$$\hat{Y}(t) = \sum_{j=1}^{N(t)} M_j, \tag{3.6}$$

where $N(t)$ is a Poisson process with rate $\lambda > 0$, and the random variables M_j are i.i.d, independent from the process N, with reflected normal density

$$f(x) = \frac{\sqrt{2}\exp\left(-\frac{x^2}{2\sigma^2}\right)}{\sigma\sqrt{\pi}}, \quad x > 0. \tag{3.7}$$

Geman, et al. [11] considered the log price process defined as

$$Y(t) = \hat{Y}_1 - \hat{Y}_2, \tag{3.8}$$

where $\hat{Y}_1, \hat{Y}_2$ are independent copies of $\hat{Y}$. They proved that Y can be defined as a time changed Brownian motion through the following construction:

$$Y(t) = \sigma B(N_1(t) + N_2(t)), \tag{3.9}$$

where B is a standard Brownian motion, N_1 and N_2 are two independent Poisson processes with the same arrival rate λ and $N_1 + N_2$ is a Poisson process with rate 2λ ($N_1 + N_2 \sim Poisson(2\lambda)$).

In order to extend the compound Poisson construction to multivariate subordination, we now specify the subordinator $\boldsymbol{G}$ defined by (2.1), so that the resulting multivariate log price model has compound Poisson margins, as in (3.9). Let $X_i \sim Poisson(2\lambda_i - a)$, $i = 1, \ldots, n$ and $Z \sim Poisson(a)$, where $0 < a < 2\lambda_j, j = 1, \ldots, n$. It follows that $X_i + Z \sim Poisson(2\lambda_i)$. Define $\boldsymbol{W}$ as in (2.4), and choose unit weighting parameters $\alpha_i = 1, i = 1, \ldots, n$. Let $\boldsymbol{G}$ be as (2.1). In this way the marginal process G_j is compound Poisson with parameter $2\lambda_j$:

$$\mathcal{L}(G_j(t)) = Poisson(2\lambda_j t), \quad j = 1, \ldots, n.$$

Using (2.3), the characteristic function of $\boldsymbol{G}(1)$ is

$$\psi_{\boldsymbol{G}}(\boldsymbol{u}) = \exp\left(\sum_{j=1}^{n}(2\lambda_i - a)(\exp(\mathrm{i}u_j) - 1)\right) + \left(a\left(\exp\left(\mathrm{i}\sum_{j=1}^{n}\alpha_j uj\right) - 1\right)\right). \tag{3.10}$$

The log price process $\boldsymbol{Y}$ is defined to have the same marginal processes considered in [11]. Therefore we impose $\mu_j = 0$, $j = 1, \ldots, n$ in the construction of Section 3, namely

$$\boldsymbol{Y}(t) = \begin{pmatrix} \sigma_1 B_1(G_1(t)) \\ \cdots \\ \sigma_n B_n(G_n(t)) \end{pmatrix}. \tag{3.11}$$

We are able to provide its Lévy triplet, as explained in Section 1. Moreover its characteristic function at time one is the following:

$$\psi_{\boldsymbol{Y}}(\boldsymbol{u}) = \exp\left(\sum_{j=1}^{n}(2\lambda_i - a)\left(\exp\left(-\mathrm{i}\frac{1}{2}\sigma_j^2 u_j^2\right) - 1\right)\right) + a\left(\exp\left(-\mathrm{i}\sum_{j=1}^{n}\alpha_j\frac{1}{2}\sigma_j^2 u_j^2\right) - 1\right).$$

The process has finite activity, because its margins do.

3.2. Normal inverse Gaussian margins

Barndorff-Nielsen [12] construct a normal inverse Gaussian process by subordination of a Brownian motion using an inverse Gaussian subordinator G. This subordinator belongs to the tempered stable family (see Appendix B). They let $\{B(t), t \geq 0\}$ be a standard Brownian motion and $\{G(t), t \geq 0\}$ be an IG process with parameters $a = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$, such that $\alpha > 0$, $-\alpha < \beta < \alpha$, $\delta > 0$. The process

$$Y(t) = \beta\delta^2 G(t) + \delta B(t), \tag{3.12}$$

is a NIG process with parameters (α, β, δ).

We adopt our construction to define a multidimensional time changed Brownian motion of NIG type.

We assume that the subordinator $\boldsymbol{G}$ defined by (2.1) has IG margins: define $X_i \sim IG(1 - a\gamma_i, \frac{b}{\gamma_i})$, $i = 1, \ldots, n$ and $Z \sim IG(a, b)$. The IG distribution is tempered stable: it follows that $\gamma_i^2 Z \sim IG(a\gamma_i, \frac{b}{\gamma_i})$. Since the marginal distributions must have nonnegative parameters, the following constraints must be satisfied:

$$b > 0, \quad 0 < a < \frac{1}{\gamma_i}, \quad i = 1, \ldots, n. \tag{3.13}$$

From the closure properties of the IG distribution it follows that $X_i + \gamma_i^2 Z$ is IG; from independence among the processes X_j, $j = 1, \ldots, n$ and Z it follows that its characteristic function is

$$\begin{aligned}\psi_{X_i+\gamma_i^2 Z} &= \exp\left(-\gamma_i a\left(\sqrt{-2\mathrm{i}u + \left(\frac{b}{\gamma_i}\right)^2} - \frac{b}{\gamma_i}\right)\right)\exp\left(-(1-a\gamma_i)\left(\sqrt{-2\mathrm{i}u + \left(\frac{b}{\gamma_i}\right)^2} - \frac{b}{\gamma_i}\right)\right)\\ &= \exp\left(-\left(\sqrt{-2\mathrm{i}u + \left(\frac{b}{\gamma_i}\right)^2} - \frac{b}{\gamma_i}\right)\right).\end{aligned} \tag{3.14}$$

Therefore: $X_i + \gamma_i^2 Z \sim IG(1, \frac{b}{\gamma_i})$. Let $\boldsymbol{W}$ be as in (2.4) and choose as weighting parameters $\alpha_i = \gamma_i^2$, $i = 1, \ldots, n$. Let $\boldsymbol{G}$ be as in (2.1). In this way the marginal process G_j is IG with parameters t and $\frac{b}{\gamma_j}$

$$\mathcal{L}(G_j(t)) = IG\left(t, \frac{b}{\gamma_j}\right), \quad j = 1, \ldots, n.$$

The characteristic function of $\boldsymbol{G}(1)$ is $\psi_{\boldsymbol{G}}(\boldsymbol{u}) = \psi_{X_i+\gamma_i^2 Z}$.

We now impose some constraints on the parameters which make the subordinated process have NIG margins. Let $\alpha_j, \beta_j, \delta_j$ be such that $\alpha_j > 0, -\alpha_j < \beta < \alpha_j, \delta > 0$. In order to get NIG margins we choose the parameter of the subordinator so that $b_j = \frac{b}{\gamma_j} = \delta_j\sqrt{\alpha_j^2 - \beta_j^2}$. Furthermore, we define the independent Brownian motions $B_j(t) = \beta_j\delta_j^2 t + \delta_j B_j(t)$, $j = 1, \ldots, n$, according to (3.12).

In accordance to the general construction of the previous section, we form the process $\boldsymbol{Y} = \{\boldsymbol{Y}(t), t > 0\}$ by time changing independent Brownian motions:

$$\boldsymbol{Y}(t) = \begin{pmatrix}\beta_1\delta_1^2 G_1(t) + \delta_1 B_1(G_1(t))\\ \cdots\\ \beta_n\delta_n^2 G_n(t) + \delta_n B_n(G_n(t))\end{pmatrix}. \tag{3.15}$$

The process $\boldsymbol{Y}$ defined in (3.15) is a Lévy process with NIG margins. Its Lévy triplet $(\boldsymbol{\gamma}_{\boldsymbol{Y}}, \Sigma_{\boldsymbol{Y}}, \nu_{\boldsymbol{Y}})$ can be derived from (3.3). Its characteristic function at time one is the following:

$$\begin{aligned}\psi_{\boldsymbol{Y}}(\boldsymbol{u}) = \exp\Bigg[&-\sum_{j=1}^{n}(1-a\gamma_j)\left(\sqrt{-2\mathrm{i}\left(\mathrm{i}\beta_j\delta_j^2 u_j - \frac{1}{2}\delta_j^2 u_j^2\right) + \frac{b^2}{\gamma_j^2}} - \frac{b}{\gamma_j}\right)\\ &- a\gamma_j\left(\sqrt{-2\mathrm{i}\sum_{j=1}^{n}\gamma_j\left(\mathrm{i}\beta_j\delta_j^2 u_j - \frac{1}{2}\delta_j^2 u_j^2\right) + \frac{b^2}{\gamma_j^2}} - \frac{b}{\gamma_j}\right)\Bigg].\end{aligned} \tag{3.16}$$

It has unbounded variation, since the marginal processes do.

3.3. Variance gamma margins

Another example of multivariate subordinator with the features of Section 2 above is the $\boldsymbol{\alpha}$-gamma process introduced in [9], that leads to a log price model with variance gamma (VG) margins. The $\boldsymbol{\alpha}$-gamma process is a generalization of the multivariate VG process introduced for the symmetric case in [13] and calibrated in [14]. The latter process was constructed by subordination of a multivariate Brownian motion $\boldsymbol{B}$ using a common gamma subordinator. The starting point is the univariate VG model Y, which is constructed as follows: let $\{B(t), t \geq 0\}$ be a standard Brownian motion, $\{G(t), t \geq 0\}$ be a gamma process with parameters $(\frac{1}{\nu}, \frac{1}{\nu})$ and let $\sigma > 0$, μ be real parameters. Then the real process Y is defined as

$$Y(t) = \mu G(t) + \sigma B(G(t)).$$

The multivariate VG is obtained by extending the previous construction considering n independent Brownian motions subordinated by a common gamma process.

The $\boldsymbol{\alpha}$-gamma process instead is constructed by time change as follows: consider $a, b, \alpha_j, j = 1, \ldots, n$ real parameters. In order to have marginal distributions with nonnegative parameters, let them satisfy the constraints

$$0 < \alpha_j < \frac{b}{a} \quad j = 1, \ldots, n. \tag{3.17}$$

Let $\mathcal{L}(X_j) = \Gamma\left(\frac{b}{\alpha_j} - a, \frac{b}{\alpha_j}\right)$ and $\mathcal{L}(Z) = \Gamma(a, b)$; assume that $X_j, j = 1, \ldots, n$ and Z are independent random variables; the random vector $\boldsymbol{W}$ defined in (2.4) satisfies $\mathcal{L}(W_j) = \Gamma(\frac{b}{\alpha_j}, \frac{b}{\alpha_j}), j = 1, \ldots, n$.

The Lévy process $\boldsymbol{G} = \{\boldsymbol{G}(t), t \geq 0\}$ associated to the distribution of $\boldsymbol{W}$,

$$\mathcal{L}(G_j(t)) = \Gamma\left(\frac{tb}{\alpha_j}, \frac{b}{\alpha_j}\right), \quad j = 1, \ldots, n,$$

is a subordinator.

The Lévy triplet of $\boldsymbol{Y}$, $(\boldsymbol{\gamma}_{\boldsymbol{Y}}, \Sigma_{\boldsymbol{Y}}, \nu_{\boldsymbol{Y}})$ is given by (3.3). Its characteristic function is

$$\psi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) = \prod_{j=1}^{n}\left(1 - \frac{\alpha_j\left(i\mu_j u_j - \frac{1}{2}\sigma_j^2 u_j^2\right)}{b}\right)^{-t\left(\frac{b}{\alpha_j} - a\right)}\left(1 - \frac{\sum_{j=1}^{n}\alpha_n(i\mu_j u_j - \frac{1}{2}\sigma_j^2 u_j^2)}{b}\right)^{-ta}. \tag{3.18}$$

The $\boldsymbol{\alpha}$-VG process has infinite activity and bounded variation, as one can show from the properties of its components.

3.4. CGMY margins

Madan and Yor [15] proved that the CGMY process, first introduced in [16], can be constructed as a time changed Brownian motion.

Let Y be a $CGMY(c, g, m, y)$ process, with parameters $c, g, m > 0$ and $y < 2$. Let us consider the stable subordinator $G' \sim S_{\frac{y}{2}}(K, \gamma)$, with Lévy measure

$$\nu'(\mathrm{d}x) = \frac{K}{x^{1+\frac{y}{2}}}\mathrm{d}x. \tag{3.19}$$

Define as Γ_k the gamma random variable with law $\Gamma(k, 1)$, and as $\Gamma(k)$ the gamma function at k.

Madan and Yor [15] take G as a subordinator absolutely continuous with respect to G', with density

$$f(z) = \mathrm{e}^{-\frac{(B^2 - A^2)z}{2}} E\left[\exp\left\{-\frac{B^2 z}{2}\frac{\Gamma_{y/2}}{\Gamma_{1/2}}\right\}\right], \tag{3.20}$$

where

$$A = \frac{g - m}{2}, \qquad B = \frac{g + m}{2}, \qquad K = \frac{c\Gamma(y/4)\Gamma(1 - y/4)}{2\Gamma(1 + y/2)}. \tag{3.21}$$

Let us denote their subordinator as $Su(c, g, m, y)$. They then define the process Y by the following

$$Y(t) = \frac{g - m}{2}G(t) + B(G(t)). \tag{3.22}$$

We now construct a multivariate subordinator of the type introduced in Section 1 so as to obtain a multivariate Lévy model with $CGMY(c_j, g_j, m_j, y)$ margins, where $c_j, g_j, m_j > 0, y < 2$.

Let $Z \sim Su(c', g, m, y)$, where $c', g, m > 0$ and $y < 2$. Let also $X_j \sim Su(c_j'', g_j, m_j, y)$, where $c_j'' > 0, g_j = \frac{g'}{\sqrt{\alpha_j}}, m_j = \frac{m'}{\sqrt{\alpha_j}}$; then $\boldsymbol{G}$ has marginal processes $G_j \sim Su(c_j g_j m_j y)$, with $c_j = c_j' + c_j''$ and $c_j' = c'\alpha_j^{y/2}$.[2]

In accordance to the general construction of the previous section, define the process $\boldsymbol{Y} = \{\boldsymbol{Y}(t), t > 0\}$ by time changing n independent Brownian motions:

$$\boldsymbol{Y}(t) = \begin{pmatrix} \frac{g_1 - m_1}{2}G_1(t) + B_1(G_1(t)) \\ \cdots \\ \frac{g_n - m_n}{2}G_n(t) + B_n(G_n(t)) \end{pmatrix}. \tag{3.23}$$

[2] In fact if $Z \sim Su(c', g, m, y)$ then $\alpha_j Z \sim Su(c_j', g_j, m_j, y)$ where $g_j = \frac{g'}{\sqrt{\alpha_j}}, m_j = \frac{m'}{\sqrt{\alpha_j}}$ and $c_j' = c'\alpha_j^{y/2}$.

The process $\boldsymbol{Y}$ is a Lévy process with CGMY margins with parameters c_j, m_j, g_j, y. Since the subordinator has zero drift its L évy triplet $(\boldsymbol{\gamma_Y}, \Sigma_{\boldsymbol{Y}}, \nu_{\boldsymbol{Y}})$ can be derived from (3.3). The variations of $\boldsymbol{Y}$ depend[3] on the parameter Y.

4. Dependence

This section is devoted to discussing the dependence structure of the above models. The subordinated Lévy model $\boldsymbol{Y}$ has nonlinear dependence. We observe that the process has dependent margins also in the symmetric case (case in which we prove that $\rho = 0$): indeed the Lévy measure of $\boldsymbol{Y}$ is given by

$$\nu_{\boldsymbol{Y}}(B) = \int_{\mathbb{R}_+^n} \rho_{\boldsymbol{s}}(B)\nu_{\boldsymbol{G}}(d\boldsymbol{s}). \tag{4.1}$$

From the expression of $\nu_{\boldsymbol{G}}$ it follows that the components of $\boldsymbol{Y}$ may jump together. Thus the processes $\sigma_j B_j(G_j(t))$ have nonlinear dependence, unless the random variable Z is degenerate.

We now turn to linear dependence, which can be useful in order to calibrate the previous models. We spend some words about linear correlation for the multivariate time changes and time changed processes of Sections 2 and 3 as a whole. In Section 6 we will specify it for the CP, NIG and VG cases considered above.

We start from the correlation matrix $\rho_{\boldsymbol{G}(t)} = (\rho_{\boldsymbol{G}(t)}(l,j))$ of the subordinator. Since

$$\mathrm{Cov}(G_l(t), G_j(t)) = \alpha_l\alpha_j V(Z(t)) \quad \text{and} \quad V(G_j(t)) = V(X_j(t)) + \alpha^2 V(Z(t)), \tag{4.2}$$

where $V(G_j(t))$ stands for the variance of $G_j(t)$, we have

$$\rho_{\boldsymbol{G}(t)}(l,j) = \frac{\alpha_l\alpha_j V(Z(t))}{\sqrt{[V(X_l(t)) + \alpha_l^2 V(Z(t))][V(X_j(t)) + \alpha_j^2 V(Z(t))]}}.$$

As concerns the subordinated process $\boldsymbol{Y}$, the variance of $Y_j(t)$ is:

$$V[Y_j(t)] = E[V[Y_j(t)|G_j(t)]] + V[E[Y_j(t)|G_j(t)]] = \sigma_j^2 E[G_j(t)] + \mu_j^2 V[G_j(t)]. \tag{4.3}$$

The lj covariance of the process at time t is:

$$\mathrm{cov}[Y_l(t), Y_j(t)] = \mu_l\mu_j \mathrm{cov}[G_1(t), G_2(t)] = \mu_l\mu_j\alpha_l\alpha_j V(Z(t)).$$

Therefore the linear correlation coefficients are

$$\rho_{\boldsymbol{Y}(t)}(l,j) = \frac{\mu_l\mu_j\alpha_l\alpha_j V(Z(t))}{\sqrt{V(Y_l(t))V(Y_j(t))}}.$$

Since all the processes involved are Lévy ones, by infinite divisibility $V(Z(t)) = V(Z)t, V(Y_j(t)) = V(Y_j(1))t, j = 1, \ldots, n$ and $\rho_{\boldsymbol{Y}(t)}(l,j)$ is independent from t.

Observe that both linear correlations $\rho_{\boldsymbol{G}(t)}$ and $\rho_{\boldsymbol{Y}(t)}$ only depend on the marginal parameters and on the variance of the subordinator's common factor Z. Therefore in order to fit both margins and correlation it is sufficient to have one spare parameter in the distribution of Z. In order to recover well known processes for representation of single asset returns we consider different specifications for the process Z. When it is a process depending on two or more parameters (such as the gamma process), we can fix all except one of them to simplify the presentation.

We now list the dependence features of the model considering advantages and drawbacks. The advantages of the $\boldsymbol{Y}$ model are:

1. each marginal distribution has its own parameters;
2. linear correlation can be fitted, and a single additional parameter is necessary to that aim;
3. it is possible to model independence.[4]

These three features cannot be captured by the standard multivariate time changed models with a univariate subordinator. Consider for instance the VG case: all the margins have a common parameter, correlations depends on the marginal parameters only, independence cannot be modelled. On the other hand the limits of the model are:

1. for given marginal parameters the model could be unable to reach very high correlation. In fact the common parameter has to satisfy some constraints that depend on the marginal parameters;

[3] If $y < 1$ the path have bounded variation, if $y \in [1, 2)$ they have unbounded variation. Moreover if $y < 0$ the process has also finite activity. In fact the marginal y_j are CGMY processes and they have finite activity if $y < 0$. Since the Lévy measures of G_j and X_j only differ for constant terms, also the Lévy measures of the subordinated processes $Y_j = B_j(G_j)$ and $B_j(X_j)$ only differ for constant terms. Thus, if $Y < 0$ the margins Y_j have finite activity then $B_j(X_j)$ have finite activity that implies (see Appendix B) $\boldsymbol{Y}$ has finite activity.

[4] Under the conditions $\mu_j > 0$ and $\alpha_j > 0, j = 1, \ldots, n$, $\rho_{\boldsymbol{Y}(t)}(l,j) = 0$ iff $V[Z(t)] = 0$, that is Z is degenerate iff the margins are independent. The process $\boldsymbol{Y}$ is a mixture of independent processes and has independent margins.

2. the process correlation is bounded above by the subordinator one: $\rho_{\mathbf{Y}(t)}(l,j) \leq \rho_{\mathbf{G}(t)}(l,j)$.[5] Equality may hold only if $\sigma_i = \sigma_j = 0$, i.e. if the Brownian components degenerate;
3. the return correlation is zero in the symmetric case, i.e. $\mu_i = \mu_j = 0$, even if the margins are uncorrelated because $V(Z(t)) \neq 0$.
4. the sign of each return correlation coefficient ρ_{jl} depends only on the sign of the product $\mu_j\mu_l$. This means that for given margins we cannot capture both negative and positive correlations.

The above drawbacks characterize also to the models constructed by a univariate time change. In that case it is possible to increase the range of dependence using correlated Brownian motions instead of independent ones. The same device cannot be adopted for $\mathbf{Y}$ if we want to remain in the Lévy class. Eberlein and Madan [8] adopt it in presence of independent subordinators. We recover their case if the common component of trade Z is degenerate.

An alternative possibility is to consider a linear transformation of $\mathbf{Y}$. The resulting process is Lévy, but it does not preserve the split of each change of time into a common and an idiosyncratic component which, following economic intuition, have been used in its construction. In order to improve the dependence features and satisfy the above intuition, in the next section we will consider a more general construction.

5. A more general model

The generalization is based on the following decomposition, which is proven in [10]:

$$\mathbf{Y} =_d \mathbf{Y}^X + \mathbf{Y}^{\alpha Z}, \tag{5.1}$$

where $\mathbf{Y}^X = (B_1(X_1), \dots, B_n(X_n))^{\mathrm{T}}$ and $\mathbf{Y}^{\alpha Z} = (B_1(\alpha_1 Z), \dots, B_n(\alpha_n Z))^{\mathrm{T}}$ are multidimensional time changed Brownian motions. $\mathbf{Y}^X$ is time changed with independent subordinators $\mathbf{X}(t)$ and therefore has independent components. $\mathbf{Y}^{\alpha Z}$ is time changed by a unique subordinator $(Z(t))$ and therefore has dependent components. $\mathbf{Y}^X$ and $\mathbf{Y}^{\alpha Z}$ are independent.

The above decomposition of $\mathbf{Y}$ provides a method to correlate the Brownian motions, remain in the Lévy setting and preserve the time change split. We consider correlated Brownian motion in the $\mathbf{Y}^Z$ component. Formally, let $\mathbf{Y}^X$ be as above and let $\mathbf{B}^\rho$ be a multidimensional Brownian motion with drift $\mu_j\alpha_j$, correlations ρ_{lj} and diffusions $\sigma_j\sqrt{\alpha_j}$. Let $\mathbf{Y}^Z_\rho$ be a time changed Brownian motion with a common subordinator, $\mathbf{Y}^Z_\rho = \mathbf{B}^\rho(Z(t))$.

Define the $\mathbb{R}^n$ valued log price process $\mathbf{Y}_\rho = \{\mathbf{Y}_\rho(t), t \geq 0\}$ as:

$$\mathbf{Y}_\rho = \mathbf{Y}^X + \mathbf{Y}^Z_\rho. \tag{5.2}$$

The process $\mathbf{Y}_\rho$ is a Lévy process, since it is the sum of two independent Lévy processes. Its characteristic function can be easily found as $\psi_A(\mathbf{z}) = \psi_{\mathbf{Y}^X}(\mathbf{z})\psi_{\mathbf{Y}^Z_\rho}(\mathbf{z})$.

Theorem 5.1. *The process $\mathbf{Y}_\rho$ defined in (5.2) has the same marginal processes of $\mathbf{Y}$ (in law).*

Proof. Let $\mathbf{Y}$ be the process in (3.2). Fix any $(\mu_j, \sigma_j, \alpha_j)$ and let F_j be the marginal distribution of the vector $\mathbf{Y} = \mathbf{Y}(1)$. From (5.1), $\mathbf{Y} =_d \mathbf{Y}^X + \mathbf{Y}^{\alpha Z}$ where $\mathbf{Y}^X$ and $\mathbf{Y}^{\alpha Z}$ are independent. Since the margins of a convolution are the convolution of the margins, we get that the convolution of $\mathcal{L}(B_j(X_j))$ and $\mathcal{L}(B_j(\alpha_j Z))$ is F_j.

Since also the processes $\mathbf{Y}^X$, $\mathbf{Y}^Z_\rho$ are independent, the law of $\mathbf{Y}_\rho$ is the convolution of their laws, and its marginal distributions are the convolutions of the marginal ones of $\mathbf{Y}^X$ and $\mathbf{Y}^Z_\rho$. Therefore we only need to verify that $\mathbf{Y}^Z_\rho$ has the same marginal distribution of $\mathbf{Y}^{\alpha Z}$.

Let us consider the marginal distribution of $\mathbf{Y}^Z_\rho$.

$$\begin{aligned} \mathcal{L}({Y_\rho}_j^Z(t)) &= \mathcal{L}(\mu_j\alpha_j Z(t) + \sigma_j\sqrt{\alpha_j}B(Z(t))) \\ &= \mathcal{L}(\mu_j\alpha_j Z(t) + \sigma_j B_j(\alpha_j Z(t))) \\ &= \mathcal{L}(Y_j^{\alpha Z}), \end{aligned} \tag{5.3}$$

where the first equality follows by construction and the scaling property of the Brownian motion. □

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5 Indeed, if $\sigma_i, \sigma_j \neq 0$,

$$\frac{\mu_1\mu_2}{\sqrt{V(Y_j)V(Y_i)}} = \frac{\mu_1\mu_2}{\sqrt{(\sigma_1^2 + \mu_1^2 V(G_j))(\sigma_2^2 + \mu_2^2 V(G_i))}} < \frac{\mu_1\mu_2}{\sqrt{\mu_1^2 V(G_j)\mu_2^2 V(G_i)}} \leq \frac{1}{\sqrt{V(G_j)V(G_i)}}$$

implies

$$\frac{\mu_1\mu_2\alpha_j\alpha_i V(Z)}{\sqrt{V(Y_j)V(Y_i)}} < \frac{\alpha_j\alpha_i V(Z)}{\sqrt{(V(G_j))(V(G_i))}}. \tag{4.4}$$

Therefore

$$\rho_{\mathbf{Y}(t)}(l,j) < \rho_{\mathbf{G}(t)}(l,j). \tag{4.5}$$

The above theorem guarantees that if $\boldsymbol{Y}$ has $CP(\sigma_j, \lambda_j)$, $VG(\mu_j, \sigma_j, \alpha_j)$, $NIG(\alpha_j, \beta_j, \delta_j)$ margins, the process $\boldsymbol{Y}_\rho$ has too.

Remark 2. Luciano and Semeraro [10] introduced the following construction for the GH case. Let $\tilde{\boldsymbol{B}}(t) = (\theta_1 + \tilde{\sigma}_1 B_1(t), \dots, \theta_n + \tilde{\sigma}_n B_n(t))$ be a Brownian motion with independent components. Let us consider a $n \times n$ matrix $A = (a_{lj})$ and define:

$$\hat{\boldsymbol{B}} = (\hat{B}_1(t), \dots, \hat{B}_n(t))^{\mathrm{T}} = A\tilde{\boldsymbol{B}}. \tag{5.4}$$

Then $\hat{\boldsymbol{B}}$ is a correlated n-dimensional Brownian motion with drift $\hat{\boldsymbol{\theta}} = A\boldsymbol{\theta}$ and diffusion matrix $\hat{\Sigma} = A\tilde{\Sigma}A^{\mathrm{T}}$, where $\tilde{\Sigma}$ is the diffusion matrix of $\tilde{\boldsymbol{B}}$.

Let us consider a n-dimensional time changed Brownian motion with one common subordinator:

$$\boldsymbol{Y}^{\boldsymbol{Z}} = (\theta_1 Z(t) + \tilde{\sigma}_1 B_1(Z(t)), \dots, \theta_n Z(t) + \tilde{\sigma}_n B_n(Z(t)))^{\mathrm{T}}.$$

Since $\boldsymbol{Y}^{\boldsymbol{Z}}$ is a Lévy process, by means of the linear transformation A, $A\boldsymbol{Y}^{\boldsymbol{Z}}$ is an $\mathbb{R}^n$ valued Lévy process (see [17]).

Define the $\mathbb{R}^n$ valued log price process $\boldsymbol{Y}_A = \{\boldsymbol{Y}_A(t), t \geq 0\}$ as:

$$\boldsymbol{Y}_A = \boldsymbol{Y}^X + A\boldsymbol{Y}^Z. \tag{5.5}$$

The process $\boldsymbol{Y}_A$ is a Lévy process, since it is the sum of two independent Lévy processes. Since $\mathcal{L}(\boldsymbol{B}^\rho(Z(t))) = \mathcal{L}(A\boldsymbol{Y}^Z(t))$, then $\mathcal{L}(\boldsymbol{Y}_A) = \mathcal{L}(\boldsymbol{Y}_\rho)$.

The α-VG specification the process $\boldsymbol{Y}_\rho$ is the convolution of a multidimensional VG with independent margins and a VG with a common gamma subordinator. A similar model for stock returns has been recently formulated in [18].[6]

The linear correlation coefficients of the process $\boldsymbol{Y}_\rho$ are:

$$\begin{aligned}\rho_{\mathbf{Y}_\rho}(l,j) &= \frac{\operatorname{cov}((AY^Z)_l, (AY^Z)_j)}{\sqrt{V(Y_l)V(Y_j)}} \\ &= \frac{\rho_{lj}\sigma_l\sigma_j\sqrt{\alpha_l}\sqrt{\alpha_j}E[Z] + \mu_l\mu_j\alpha_l\alpha_j V(Z)}{\sqrt{V(Y_l)V(Y_j)}} = \frac{\rho_{lj}\sigma_l\sigma_j\sqrt{\alpha_l}\sqrt{\alpha_j}E[Z]}{\sqrt{V(Y_l)V(Y_j)}} + \rho_{\mathbf{Y}}(l,j).\end{aligned} \tag{5.6}$$

The correlations of $\boldsymbol{Y}_\rho$ have an additional term with respect to the correlations of $\boldsymbol{Y}$ but are still independent of time. The correlations $\rho_{\mathbf{Y}_\rho}$ may be greater or smaller than $\rho_{\mathbf{Y}}$, depending on the sign of ρ_{lj}.

First of all we observe that the correlation coefficients depend on two moments of the common component. In this case we have two spare parameters of the common time change in order to improve the fit of the correlation.

The above correlations show that $\boldsymbol{Y}_\rho$ allows to overcome the limits of the model $\boldsymbol{Y}$.

1. Differently from the independent Brownian motion case, the correlation $\rho_{\mathbf{Y}_\rho}$ is affected by the Brownian motion correlation, which is unrelated to the margins. This means that, for given marginal distributions, correlation can be increased up to the maximum level $\rho_{ij} = 1$;
2. $\rho_{\mathbf{Y}}(l,j)$ can be equal to $\rho_{\boldsymbol{G}}(l,j)$ also in a non-degenerate case: we provide an example for the VG case;
3. The correlation can be different from zero also in the symmetric case;
4. it is possible to have negative return correlation also if $\mu_l\mu_j > 0$;

Nonetheless, the process $\boldsymbol{Y}_\rho$ has many more parameters than $\boldsymbol{Y}$ (all the ρ_{lj}) that is why we kept both models in our presentation, and will calibrate both in the example.

6. Linear correlation

In this section we specify the linear correlation coefficients for the cases VG, CP and NIG. We consider both the models with independent Brownian motions and the models with correlated Brownian motions.

6.1. Compound Poisson

Consider the independent Brownian motion model $\boldsymbol{Y}$. The linear correlation coefficients of the process are:

$$\rho_{\mathbf{Y}(t)}(l,j) = \frac{\mu_l\mu_j a}{2\sqrt{\lambda_l(\sigma_l^2 + \mu_l^2)\lambda_j(\sigma_j^2 + \mu_j^2)}}.$$

If we focus on the parametrization in [11], in which $\mu_j = 0, j = 1, \dots, n$, the linear correlation of $\boldsymbol{Y}$ is zero, while we can capture nonlinear dependence. Indeed if $a \neq 0$, then $V[Z(t)] = at \neq 0$, the correlation of the subordinator is different from

[6] They also recognize that it can be extended to other Lévy processes.

zero and the margins of $\boldsymbol{Y}$ are positively associated. Moreover we have independence if $a \to 0$ and maximal dependence, that corresponds to maximal correlation for the subordinator, if $a \to 2\lambda_j$ for each $j = 1, \dots, n$; in the last case $\boldsymbol{G}$ is a.s. a univariate subordinator.

If we consider the more general process $\boldsymbol{Y}_\rho$ with Poisson margins the linear correlation coefficients are

$$\rho_{\boldsymbol{Y}_\rho(t)}(l,j) = \frac{\rho_{lj}\sigma_l\sigma_j a + \mu_l\mu_j a}{2\sqrt{\lambda_l(\sigma_l^2+\mu_l^2)\lambda_j(\sigma_j^2+\mu_j^2)}}.$$

6.2. Normal inverse Gaussian

We now consider the NIG independent Brownian motion model. The linear correlations of the subordinator are

$$\rho_{\boldsymbol{G(t)}}(l,j) = \frac{\gamma_l^2\gamma_j^2\frac{a}{b^3}}{\sqrt{\left[\frac{(1-a\gamma_l)\gamma_l^3}{b^3}+\gamma_l^2\frac{a}{b^3}\right]\left[\frac{(1-a\gamma_j)\gamma_j^3}{b^3}+\gamma_j^2\frac{a}{b^3}\right]}}.$$

Observe that $\rho_{\boldsymbol{G(t)}}(l,j) = 1$ if $a = \frac{1}{\gamma_j} = \frac{1}{\gamma}, j = 1, \dots, n$ (in this way $\rho_{\boldsymbol{G(t)}(l,j)} = \gamma$) and $\gamma = 1$. By so doing we obtain the subcase with one subordinator Z. Its law becomes $IG(1, b)$.

The linear correlation coefficients of the subordinated process at time t are:

$$\begin{aligned}\rho_{\boldsymbol{Y}(t)}(l,j) &= \frac{\beta_l\delta_l^2\beta_j\delta_j^2\gamma_l^2\gamma_j^2\frac{a}{b^2}}{\sqrt{\left(\delta_l^2\gamma_l+\frac{\beta_l^2\delta_l^4\gamma_l^3}{b^2}\right)\left(\delta_j^2\gamma_j+\frac{\beta_j^2\delta_j^4\gamma_j^3}{b^2}\right)}}\\ &= \frac{\beta_l\delta_l^2\beta_j\delta_j^2\frac{\gamma_l^2}{b^2}\frac{\gamma_j^2}{b^2}a}{\frac{1}{b}\sqrt{\left(\delta_l^2\frac{\gamma_l}{b}+\beta_l^2\delta_l^4\frac{\gamma_l^3}{b^3}\right)\left(\delta_j^2\frac{\gamma_j}{b}+\beta_j^2\delta_j^4\frac{\gamma_j^3}{b^3}\right)}}.\end{aligned}$$

From this representation it is clear that in order to study the correlation the assumption $b = 1$ is not restrictive.[7]

The only way to capture independence is to let a go to zero. In order to capture the maximal dependence, we need $a = 1$ and we have one subordinator.

The linear correlations for the general $\boldsymbol{Y}_\rho$ return process for the NIG specifications are:

$$\rho_{\boldsymbol{Y}_\rho(t)}(l,j) = \frac{\beta_l\delta_l^2\beta_j\delta_j^2\gamma_l^2\gamma_j^2\frac{a}{b^2}+\rho_{lj}\delta_l\delta_j\gamma_l\gamma_j\frac{a}{b}}{\sqrt{\left(\delta_l^2\gamma_l+\frac{\beta_l^2\delta_l^4\gamma_l^3}{b^2}\right)\left(\delta_j^2\gamma_j+\frac{\beta_j^2\delta_j^4\gamma_j^3}{b^2}\right)}}.$$

6.3. α-variance gamma

The linear correlations of the $\boldsymbol{\alpha}$-gamma subordinator are increasing in α_l, α_j:

$$\rho_{\boldsymbol{G}(t)}(l,j) = \frac{a}{b}\sqrt{\alpha_l\alpha_j}.$$

The linear correlation coefficients of the independent Brownian motion process $\boldsymbol{Y}$ are:

$$\rho_{\boldsymbol{Y}(t)}(l,j) = \frac{\mu_l\mu_j\alpha_l\alpha_j\frac{a}{b^2}}{\sqrt{(\sigma_l^2+\mu_l^2\frac{\alpha_l}{b})(\sigma_j^2+\mu_j^2\frac{\alpha_j}{b})}} = \frac{\mu_l\mu_j\alpha_l\alpha_j a}{b\sqrt{(b\sigma_l^2+\mu_l^2\alpha_l)(b\sigma_j^2+\mu_j^2\alpha_j)}}.$$

The correlations of the process involve all the parameters, and for any couple of fixed marginal distributions the linear correlation is a function of a only.[8]

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7 b is a parameter of the common component $Z(t)$, whose distribution is $GIG(a, b)$. Therefore since we only fit the variance of $Z(t)$ it is not restrictive to fix $b = 1$.

8 This is the main contribution of the $\boldsymbol{\alpha}$-VG generalization with respect to VG correlation, since changing a we can modify the correlation of the process, without modifying the marginal distributions of the process. On the contrary, in the Variance Gamma process with a common gamma subordinator used in the previous literature ($\rho_{\boldsymbol{G}(t)} = 1$), for fixed parameters of the lj marginal processes, the correlation coefficient was uniquely determined.

The linear correlation coefficients for $\boldsymbol{Y}_\rho$ in the VG case are:

$$\rho_{\boldsymbol{Y}_\rho(t)}(l,j) = \frac{\rho_{lj}\sigma_l\sigma_j\sqrt{\alpha_l}\sqrt{\alpha_j}\frac{a}{b} + \mu_l\mu_j\alpha_l\alpha_j\frac{a}{b^2}}{\sqrt{(\sigma_l^2 + \mu_l^2\frac{\alpha_l}{b})(\sigma_j^2 + \mu_j^2\frac{\alpha_j}{b})}} = \frac{\rho_{lj}\sigma_l\sigma_j\sqrt{\alpha_l}\sqrt{\alpha_j}ab + \mu_l\mu_j\alpha_l\alpha_j a}{b\sqrt{(b\sigma_l^2 + \mu_l^2\alpha_l)(b\sigma_j^2 + \mu_j^2\alpha_j)}}.$$

We show that $\rho_{\boldsymbol{Y}_\rho} = \rho_{\boldsymbol{G}}$ is possible, also for $\sigma_l, \sigma_j \neq 0$. Choose $\sigma_j = \mu_j\sqrt{\alpha_j}$ for both j and i and $\rho_{ij} = 1$. We get:

$$\rho_{\boldsymbol{Y}_\rho} = \frac{\sigma_l\sigma_j\rho_{\boldsymbol{G}} + \sigma_l\sigma_j\rho_{\boldsymbol{G}}}{\sqrt{(\sigma_l^2 + \sigma_l^2)(\sigma_j^2 + \sigma_j^2)}} = \frac{2\sigma_l\sigma_j\rho_{\boldsymbol{G}}}{\sqrt{4(\sigma_l^2)(\sigma_j^2)}} = \rho_{\boldsymbol{G}}.$$

Before attempting the empirical analysis we motivate our choice to consider a simplification in the parameters. As you can easily verify, the correlations of the NIG and VG processes depend also on a parameter b. This parameter is common to all margins, since it is one of the two parameters of the time change common component. We explained above why, in the independent Brownian motion case, we can fix it without loss of generality. In the more general model, b could be fitted. We keep it equal to one in the sections to follow, since we want our calibrations for the general model to be comparable to the independent Brownian motion one.

7. Calibration and correlation flexibility

In this section we provide a calibration technique for the processes introduced above, which separates the marginal from the joint fit. We are interested in their correlation – or linear dependence – flexibility. By correlation flexibility we mean the ability to capture or match the estimated correlation in return data. Since financial returns often present positive and sometimes high correlation, we will be concerned mainly with capturing positive and high correlation.

All of the above models allow to capture independence, thanks to the presence of a multivariate – instead of univariate – subordinator. The models in which the time changed Brownian motions were independent are theoretically able to span a wide range of dependence, when the marginal parameters vary. However, in financial applications – and with our processes in particular – marginal parameters are given (from univariate derivative prices or underlying time series) when it comes to dependence calibration (from a correlation matrix). For fixed marginal parameters, such models often span a limited range of dependence. For this reason we generalized them by considering correlated Brownian motions. It is clear from expression that return correlations in the latter models can be greater or smaller than their counterpart in the former models. We are going to show that – for fixed marginal parameters – the latter may reach high correlations.

We provide such application on returns from seven major stock indices. For this reason, we limit the application itself to NIG and α-VG, disregarding the Poisson model. For each of them, and without loss of generality, we fix b to the value 1 (see Section 6.3). Then, we

1. calibrate the marginal parameters;
2. choose the value of a which corresponds to maximal return dependence;
3. compute the matrix correlation for returns with independent Brownians first, with maximally dependent Brownians then;
4. compare the maximal dependencies in the two cases with the sample correlation matrix.[9]

We used as raw data the Bloomberg closing prices and the quotes of the call options on seven stock indices: NASDAQ, CAC 40, FTSE 100, S & P 500, DAX, Nikkei 225, Hang Seng. The options had three months to expiry. For each index, we selected six strikes (the closest to the initial price) and we monitored the corresponding option prices over a one-hundred days window, from 7/14/06 to 11/30/06. These were used in order to infer the marginal parameters, as specified below. The returns on the underlying indices were computed over the same window (via the closing quotes) and used in order to compute the sample linear correlation matrix.

In correspondence to the α-VG marginal model, we estimated the marginal parameters using our knowledge of the (marginal) characteristic function, namely (A.8). From the characteristic function, call option prices were indeed obtained using the Fractional Fast Fourier Transform (FRFT) in [19], which is more efficient than the standard FFT. In correspondence to the NIG, we adopted moment matching, to make the reader aware of the alternative possibility.

For the α-VG, we adopted the following procedure to make the marginal parameters independent of the initial guess (needed in the Fourier approach): using the option quotes of the first day only, we obtained the parameter values which minimized the mean square error between theoretical and observed prices, the theoretical ones being obtained by FRFT. We used the results as guess values for the second day, the second day results as guess values for the third day, and so on. The marginal parameters used here – and presented in Table 1 – are the average ones.

For the NIG, we computed the marginal parameters by moment matching. More precisely, we fixed them by matching the first four moments of the VG and NIG cases. The relationships between the moments and the process parameters are in Appendix A. The values so obtained are in Table 2.

The estimated, sample correlation matrix was obtained by standard calculations as in Table 3.

9 Since the risk neutral empirical correlation matrix is not available we use the historical one as a proxy for it.

Table 1
Calibrated parameters of α-VG model.

Asset	μ	σ	α
S & P	−0.6490	0.0224	0.1021
Nasdaq	−0.6730	0.1062	0.1317
CAC 40	−0.4674	0.1031	0.1109
FTSE	−0.5865	0.0450	0.0313
Nikkei	−0.3386	0.1595	0.1042
DAX	−0.2700	0.1334	0.1410
Hang Seng	−1.6790	0.0788	0.0279

Table 2
Calibrated parameters of NIG model.

Asset	α	β	δ	γ
S & P	1.0910	−0.2170	3.1740	0.2947
Nasdaq	1.1690	−0.2920	2.5850	0.3418
CAC 40	1.0670	−0.2560	2.3440	0.4119
FTSE	1.2540	−0.3930	0.8180	1.0266
Nikkei	1.2780	−0.2600	1.6120	0.4958
DAX	0.7390	−0.1220	9.9440	0.1380
Hang Seng	1.0280	−0.1480	4.0250	0.2442

Table 3
Estimated correlation matrix.

	S & P	Nasdaq	CAC 40	FTSE	Nikkei	Dax
Nasdaq	0.903					
CAC 40	0.615	0.540				
FTSE	0.550	0.443	0.854			
Nikkei	0.019		0.250	0.245		
Dax	0.643	0.583	0.950	0.220	0.827	
Hang Seng	0.034	0.077	0.254	0.526	0.239	0.234

Table 4
Maximal correlation, independent Brownian motion NIG model (parameters in parentheses).

	S & P	Nasdaq	CAC 40	FTSE	Nikkei	Dax
Nasdaq	0.047 (2.926)					
CAC 40	0.041 (2.428)	0.056 (2.428)				
FTSE	0.034 (0.974)	0.047 (0.974)	0.049 (0.974)			
Nikkei	0.032 (2.017)		0.046 (2.017)	0.047 (0.974)		
Dax	0.023 (3.394)	0.027 (2.926)	0.024 (2.428)	0.020 (0.974)	0.018 (2.017)	
Hang Seng	0.027 (3.394)	0.031 (2.926)	0.027 (2.428)	0.023 (0.974)	0.021 (2.017)	0.018 (4.095)

Please notice that one correlation coefficient (between Nikkei and Nasdaq) has been omitted. It was negative, and would have required specific treatment and comments below. Since we want to maintain general our example, we decided not to enter into the comment and calculation modifications for negative dependence. Nonetheless, the theoretical model can deal with negative correlation.

7.1. *Normal inverse Gaussian*

The maximal correlation reached by the independent Brownian motion model $\boldsymbol{Y}$ in the NIG case corresponds to $a = \min\{\frac{1}{\gamma_i}, \frac{1}{\gamma_j}\}$. For each pair of assets Table 4 gives the maximal model correlation, namely the values of ρ corresponding to such a, as well as the a value, in parentheses.

There is only one case in which the NIG model is admissible, since the model correlation matrix is greater than the sample one. The reader can get aware of this by looking at Table 5, which is the difference between the previous two.

The maximal correlation allowed by the extended, dependent Brownian motion process $\boldsymbol{Y}_\rho$ corresponds – for each pair of assets – to the above value of a and $\rho = 1$. It is presented in Table 6.

The reader can get aware of the increased ability to capture high correlation by looking at Table 7, which is the difference between Table 6 and the sample correlation matrix.

In ten cases we were then able to increase correlation and make the model able to describe actual dependence.

Table 5
Difference between the model and sample correlation matrix, independent Brownian motion NIG case.

	S & P	Nasdaq	CAC 40	FTSE	Nikkei	Dax
Nasdaq	−0.856					
CAC 40	−0.574	−0.484				
FTSE	−0.516	−0.396	−0.805			
Nikkei	0.013		−0.204	−0.198		
Dax	−0.620	−0.556	−0.926	−0.200	−0.809	
Hang Seng	−0.007	−0.046	−0.227	−0.503	−0.218	−0.216

Table 6
Maximal correlation in the extended model, NIG case.

	S & P	Nasdaq	CAC 40	FTSE	Nikkei	Dax
Nasdaq	0.927					
CAC 40	0.845	0.911				
FTSE	0.532	0.576	0.632			
Nikkei	0.771		0.911	0.691		
Dax	0.684	0.633	0.577	0.362	0.527	
Hang Seng	0.908	0.840	0.766	0.480	0.700	0.752

Table 7
Difference between the model and sample correlation matrix, extended Brownian motion NIG case.

	Nasdaq	CAC 40	FTSE	Nikkei	Dax
CAC 40	0.372				
FTSE	0.133	−0.222			
Nikkei		0.661	0.445		
Dax	0.049	−0.373	0.143	−0.299	
Hang Seng	0.763	0.512	−0.046	0.461	0.518

Table 8
Maximal correlation, independent Brownian motion α-VG model (parameters in parentheses).

	S & P	Nasdaq	CAC 40	FTSE	Nikkei	Dax
Nasdaq	0.803 (7.590)					
CAC 40	0.795 (9.020)	0.701 (7.590)				
FTSE	0.505 (9.791)	0.410 (7.590)	0.406 (9.020)			
Nikkei	0.556 (9.593)		0.457 (9.020)	0.284 (9.593)		
Dax	0.512 (7.092)	0.536 (7.092)	0.447 (7.092)	0.261 (7.092)	0.294 (7.092)	
Hang Seng	0.500 (9.791)	0.406 (7.590)	0.403 (9.020)	0.834 (31.976)	0.282 (9.593)	0.259 (7.092)

Table 9
Difference between the model and sample correlation matrix, independent Brownian motion α-VG case.

	S & P	Nasdaq	CAC 40	FTSE	Nikkei	Dax
Nasdaq	−0.100					
CAC 40	0.180	0.161				
FTSE	−0.045	−0.033	−0.448			
Nikkei	0.537		0.207	0.039		
Dax	−0.131	−0.047	−0.503	0.041	−0.533	
Hang Seng	0.466	0.329	0.149	0.308	0.043	0.025

7.2. α-variance gamma

The maximal correlation allowed by the model $\boldsymbol{Y}$ in the VG case corresponds to $a = \min\{\frac{1}{\alpha_i}, \frac{1}{\alpha_j}\}$. For each pair of assets Table 8 gives the values of ρ corresponding to the maximal theoretical correlation, as well as their a parameter:

As in Table 8, we present their differences w.r.t. the sample correlations, which are positive in a greater number of cases than before (12 instead of 1); see Table 9.

If we consider the dependent Brownian motions version, namely the process $\boldsymbol{Y}_\rho$, the maximal correlation for each pair corresponds to the values in Table 8 for a and $\rho = 1$. We get the model correlation matrix as in Table 10.

We have an improvement both with respect to the decorrelated case and to the NIG, correlated one. Table 11 presents the differences between the $\boldsymbol{Y}_\rho$ and sample correlations. It shows clearly the improvement, since it has positive entries.

Table 10
Maximal correlation in the extended model, α-VG case.

	S & P	Nasdaq	CAC 40	FTSE	Nikkei	Dax
Nasdaq	0.903					
CAC 40	0.615	0.540				
FTSE	0.550	0.443	0.854			
Nikkei	0.019		0.250	0.245		
Dax	0.643	0.583	0.950	0.220	0.827	
Hang Seng	0.034	0.077	0.254	0.526	0.239	0.234

Table 11
Difference between the model and sample correlation coefficients, extended α-VG case.

	S & P	Nasdaq	CAC 40	FTSE	Nikkei	Dax
Nasdaq	−0.062					
CAC 40	0.238	0.364				
FTSE	−0.021	0.044	−0.331			
Nikkei	0.626		0.648	0.219		
Dax	−0.057	0.260	−0.113	0.191	0.032	
Hang Seng	0.485	0.345	0.164	0.340	0.054	0.035

8. Conclusion

This paper has studied Lévy pure jump models generated by a change of time (a subordinator) with both a common and an idiosyncratic component. Such a representation was motivated by recent evidence on the factor structure of trade. It brought us to build both a more parsimonious – but less flexible in terms of high correlation – version, and a less parsimonious version, able to capture high dependence. We focused on the latter, for calibration purposes. We indeed provided an example of marginal calibration to stock market data, together with an analysis of its dependence flexibility.

The application to multivariate pricing and risk evaluation is in the agenda of future research. Our interest stems from the ability of the processes theoretically characterized in the paper to capture fat tails and skewness both at the marginal and at the joint level.

Appendix A

Here we recall the definitions of the real processes which are the basis of our multivariate generalization.

A.1. Normal inverse Gaussian

An *inverse Gaussian* (IG) process with parameters (a, b) is a Lévy process with the following characteristic function:

$$\psi_{IG}(z) = \exp t\left(-a\left(\sqrt{-2\mathrm{i}u + b^2} - b\right)\right). \tag{A.1}$$

The Lévy measure of the IG process is 1/2-stable,

$$\nu_{IG}(x) = (2\pi)^{1/2} a x^{-3/2} \exp(-1/2b^2 x)\mathbf{1}_{(0,+\infty)}(x)\mathrm{d}x. \tag{A.2}$$

A *normal inverse Gaussian* (NIG) process with parameters $\alpha > 0$, $-\alpha < \beta < \alpha$, $\delta > 0$ is a Lévy process $X_{NIG} = \{X_{NIG}(t), t \geq 0\}$ with characteristic function

$$\psi_{NIG}(z) = \exp t\left(-\delta\left(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2}\right)\right). \tag{A.3}$$

A NIG process has no Gaussian component.

The process is of infinite variation.

We end with the moments of the distribution: the mean m, the variance v, the skewness s and the kurtosis k.

$$m = \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}} \tag{A.4}$$

$$v = \alpha^2\delta\left(\alpha^2 - \beta^2\right)^{-\frac{3}{2}} \tag{A.5}$$

$$s = 3\beta\alpha^{-1}\delta^{-\frac{1}{2}}(\alpha^2 - \beta^2)^{-\frac{1}{4}} \tag{A.6}$$

$$k = 3\left(1 + \frac{\alpha^2 + 4\beta^2}{\delta\alpha^2\sqrt{\alpha^2 - \beta^2}}\right). \tag{A.7}$$

A.2. Variance gamma

A *variance gamma* process is a real Lévy process $X_{VG} = \{X_{VG}(t), t \geq 0\}$ which can be obtained as a Brownian motion with drift time changed by a gamma process.

A gamma process $\{G(t), t \geq 0\}$ with parameters (a, b) is a Lévy process so that the defining distribution of $X(1)$ is gamma with parameters (a, b) (shortly $\mathcal{L}(X(1)) = \Gamma(a, b)$). It is a finite variation Lévy process. Its Lévy triplet is

$$\gamma = \frac{a\,(1-\exp(-b))}{b},$$
$$A = 0$$
$$\nu_G(\mathrm{d}x) = a\exp(-bx)x^{-1}\mathbf{1}_{(0,+\infty)}(x)\mathrm{d}x.$$

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion, $\{G(t), t \geq 0)$ be a gamma process with parameters $(\frac{1}{\nu}, \frac{1}{\nu})$ and $\sigma > 0$, θ be real parameters; then the process X_{VG} is defined as

$$X_{VG}(t) = \theta G(t) + \sigma B(G(t)).$$

The characteristic function of X_{VG} is the following

$$\psi_{VG}(u) = \left(1 - \mathrm{i}u\theta\nu + \frac{1}{2}\sigma^2\nu u^2\right)^{-\frac{t}{\nu}}. \tag{A.8}$$

The paths of the VG process are of infinite activity and finite variation. We end with the moments of the distribution: the mean m, the variance υ, the skewness s and the kurtosis k.

$$m = \theta \tag{A.9}$$

$$\upsilon = \sigma^2 + \nu\theta^2 \tag{A.10}$$

$$s = \frac{\theta\nu(3\sigma^2 + 2\nu\theta^2)}{(\sigma^2 + \nu\theta^2)^{3/2}} \tag{A.11}$$

$$k = 3(1 + 2\nu - 4\nu\sigma^4(\sigma^2 + \nu\theta^2)^{-2}). \tag{A.12}$$

A.3. CGMY

The Carr Geman Madan Yor [16] process is a Lévy process $X_{cgmy} = \{X_{cgmy}(t), t \geq 0\}$ whose characteristic function is

$$\psi_{cgmy}(u) = \exp(Ct\Gamma(-Y)((M - \mathrm{i}u)^Y - M^Y + (G + \mathrm{i}u)^Y - G^Y)), \tag{A.13}$$

where $c, g, m > 0$ and $y < 2$. The path regularity changes for different values of the parameter y: if $y < 0$ the paths have finite activity; if $y \in [0, 1)$ they have infinite activity and finite variation; if $y \in [1, 2)$ they have infinite variation.

Appendix B

B.1. Stable subordinators

In this appendix we recall some properties of stable and tempered stable subordinators. For a complete treatment see [20].

A random variable X has stable distribution with parameters $0 < \alpha \leq 2$, $\sigma > 0$, $-1 < \beta < 1$ and $\gamma \in \mathbb{R}$, shortly $X \sim S_\alpha(\sigma, \beta, \gamma)$, if its characteristic function has the form:

$$\psi_X(z) = \begin{cases} \exp\left\{-\sigma^\alpha|z|^\alpha\left(1 - \beta(\operatorname{sign} z)\tan\dfrac{\pi\alpha}{2}\right) + \mathrm{i}\gamma z\right\}, & \alpha \neq 1 \\ \exp\left\{-\sigma|z|\left(1 + \beta(\operatorname{sign} z)\beta\dfrac{2}{\pi}ln|z|\right) + \mathrm{i}\gamma z\right\} & \alpha = 1. \end{cases} \tag{B.1}$$

Since γ affects only location, we assume $\gamma = 0$ for simplicity.

An α-stable real subordinator G is given by a stable random variable X with support $[0, \infty)$, $X \sim S_\alpha(\sigma, 1, 0)$ with $0 < \alpha < 1$.

The Lévy measure of a stable subordinator has the following expression

$$\nu_G(\mathrm{d}x) = \frac{c_G}{x^{\alpha+1}}1_{x>0}, \tag{B.2}$$

where $c_G = c(\alpha)\sigma^\alpha$, $c(\alpha) > 0$ and $1_{x>0}$ is the indicator function of the set $x > 0$.

If the subordinators X_j and Z are α-stable then $\boldsymbol{G}$ has α-stable margins. Let $X_j \sim S_\alpha(\sigma_j, 1, 0)$ and $Z \sim S_\alpha(\sigma_z, 1, 0)$, so that $\alpha_j Z \sim S_\alpha(\sigma_z \alpha_j, 1, 0)$. By Propositions 1.2.1 and 1.2.3 in [20] $X_j + \alpha_j Z$ is α-stable and its law is

$$\mathcal{L}(X_j + \alpha_j Z) = S_\alpha(\sigma_{G_j}, 1, 0), \tag{B.3}$$

where $\sigma_{G_j} = (\sigma_j^\alpha + (\sigma_z \alpha_j)^\alpha)^{1/\alpha}$.

Tempered stable subordinators, first introduced in [21], are characterized by the following Lévy measure:

$$\nu(x) = \frac{c\mathrm{e}^{-\lambda x}}{x^{\alpha+1}} 1_{x>0}. \tag{B.4}$$

Let us denote the corresponding infinitely divisible distribution by $X \sim TS_\alpha(c, \lambda)$, where $0 < \alpha < 1$, $\lambda > 0$ and $c > 0$. The distribution of the sum of two tempered stable processes, analogously to the non-tempered case, can be characterized as follows: if $X \sim TS_\alpha(c_X, \lambda)$ and $Y \sim TS_\alpha(c_Y, \lambda)$ their sum is $TS_\alpha(c_X + c_Y, \lambda)$ and $\alpha_i X \sim TS_\alpha(c_X \alpha_i^\alpha, \frac{\lambda}{\alpha_i})$. Therefore if $X_i \sim TS_\alpha(c_i, \frac{\lambda}{\alpha_i})$ for $i = 1, \ldots, n$ and $Z \sim TS_\alpha(c_z, \gamma_z, \lambda)$, then $\boldsymbol{G}$ has margins $TS_\alpha(c_i + c_z \alpha_i^\alpha, \frac{\lambda}{\alpha_i})$.

Consider a stable subordinator G_B with Lévy measure given by (B.2). A subordinator G_A is absolutely continuous with respect to G_B (see [15,17] for a more general definition), if

$$\nu_A(\mathrm{d}x) = f(x)\nu_B(\mathrm{d}x) = f(x)\frac{c_G}{x^{1+\alpha}}\mathrm{d}x \tag{B.5}$$

and

$$\int_0^\infty \nu_B(\mathrm{d}t)(\sqrt{f(t)} - 1)^2 < \infty, \tag{B.6}$$

where $f(x)$ is called the density. Obviously if X_j and Z are α-stable continuous with the same density, their sum is.

All the previous classes of subordinators are characterized by the fact that the difference between the Lévy measures of X_j, Z and G_j is a constant.

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